Chapter 3

Mathematical models

3.1 Introduction

A mathematical model is an equation which is intended to match or model the behavior of some natural quantities.

Exponential functions are found in many mathematical models. Exponential, surge and logistic models make use of exponential functions and are described in sections 3.2 to 3.4.

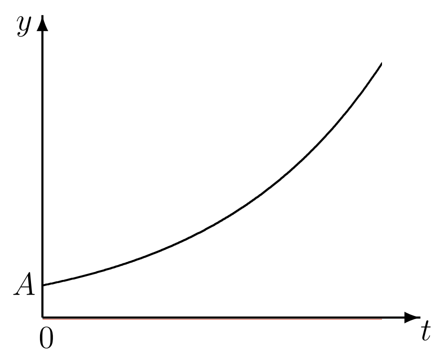
3.2 Exponential models

Exponential growth and decay models have the form

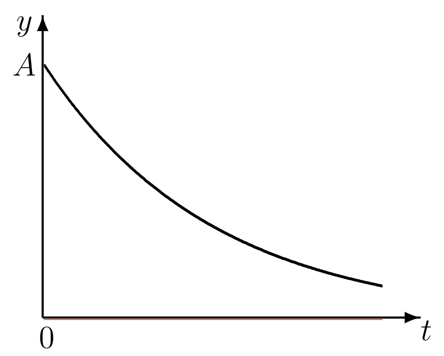
$$y = A\,e^{bt}, t \geq 0$$

for constants A and b, where independent variable t usually represents *time*.

(a) *Growth Model*: $b > 0$



(b) *Decay Model*: $b < 0$



Exponential growth models[1] are typically used to model populations that have a

[1] Also known as Malthusian models

constant percentage growth rate due to an unchanging environment.[2] Populations can range from micro-organisms to people.

Exponential decay models are typically used to model the loss of matter that has a constant percentage decay rate.[3] Examples include herbicides, radioactive materials and the elimination of medicines from the body.

Example

population growth

The population of a rabbit colony grows according to the exponential growth model

$$P(t) = 60\, e^{1.6t}$$

where time t is given in years.

This model shows that ...

- the initial population was

$$P(0) = 60\, e^{1.6\times 0} = 60 \text{ rabbits}$$

- at t years, the population grew at the rate

$$\begin{aligned} \frac{dP}{dt} &= 60 \times 1.6\, e^{1.6t} \\ &= 96\, e^{1.6t} \text{ rabbits per year} \end{aligned}$$

- the constant growth rate per head of population was

$$\begin{aligned} \frac{dP}{dt} \div P &= \frac{60 \times 1.6\, e^{1.6t}}{60\, e^{1.6t}} \\ &= 1.6 \text{ rabbits per year per head of population} \end{aligned}$$

...a growth rate of 160% per year.

The model can also be used for predictions:

(a) After 5 years there will be

$$P(5) = 60\, e^{1.6\times 5} \approx 7291 \text{ rabbits}$$

(b) The time taken for the population to reach 10,000 can be found from solving the equation $60\, e^{1.6t} = 10000$.

$$\begin{aligned} 60\, e^{1.6t} &= 10000 \\ e^{1.6t} &= 10000/60 \\ 1.6t &= \ln(10000/60) \\ t &= \frac{\ln(10000/60)}{1.6} \\ &= 3.2 \text{ years} \end{aligned}$$

[2] percentage growth rate = growth rate per head of population × 100 %
[3] percentage decay rate = decay rate per amount of material × 100 %

Example

bacterial growth

The amount of live bacteria in a Petri dish is modelled by the formula

$$M(t) = 50\,e^{0.18t} \text{ gm}$$

after t days.

You can see that

- the initial amount of live bacteria was

$$M(0) = 50\,e^{0.18\times 0} = 50 \text{ gm}$$

- the bacterial grew at the rate

$$\begin{aligned}\frac{dM}{dt} &= 50 \times 0.18\,e^{0.18t} \\ &= 9\,e^{0.18t} \text{ gm/day}\end{aligned}$$

- the constant growth rate per gram was

$$\begin{aligned}\frac{dM}{dt} \div M &= \frac{50 \times 0.18\,e^{0.18t}}{50\,e^{0.18t}} \\ &= 0.18 \text{ gm/day per gram}\end{aligned}$$

...a growth rate of 8% per day

Example

decay model

In laboratory conditions, the mass $M(t)$ of a pesticide decayed according to the exponential decay model.

$$M(t) = 10\,e^{-0.15t} \text{ gm}$$

after t days.

The model shows that

- the pesticide decayed at the rate

$$\begin{aligned}\frac{dM}{dt} &= 10 \times (-0.15)\,e^{-0.15t} \\ &= -1.5\,e^{-0.15t} \text{ gm/day}\end{aligned}$$

- the constant decay rate per gram was

$$\begin{aligned}\frac{dM}{dt} \div M &= \frac{10 \times (-0.15)\,e^{-0.15t}}{10\,e^{-0.15t}} \\ &= -0.15 \text{ gm/day per gram}\end{aligned}$$

...a decay rate of 15% per day

Newton's *Law of Cooling* models how the temperature $T(t)$ of an object changes[4] from an initial temperature of $T(0)$ when it is placed in an environment having temperature T_{env}.

$$T(t) = T_{env} + (T(0) - T_{env})e^{-kt}$$

Example

heat transfer

A turkey is cooking in a convection oven which is at a baking temperature of $200°C$. The turkey starts at a emperature of $20°C$ and after a half hour has warmed to $30°C$. How long will it take to warm to a well-done temperature of $80°C$?

Answer

We need to find k first of all. As turkey took 30 min to heat from $20°C$ to $30°C$, we have

$$\begin{aligned} 30 &= 200 + (20 - 200)e^{-0.5k} \\ e^{-0.5k} &= \frac{170}{180} \\ k &= 0.1143 \end{aligned}$$

To find the time taken to heat to $80°C$, solve

$$\begin{aligned} 80 &= 200 + (20 - 200)e^{-0.1143t} \\ e^{-0.1143t} &= \frac{120}{180} \\ t &= 3.5 \text{ hours} \end{aligned}$$

Exercise 3.2

1. A population of bacteria is given by $P(t) = 5\,000\,e^{0.18t}$ after t hours.
   (a) What is the population at
   (i) $t = 0$ hours (ii) $t = 30$ minutes (iii) $t = 2$ hours?
   (b) How long would it take for the population to reach $15\,000$?
   (c) What is the rate of increase of the population at
   (i) $t = 0$ (ii) $t = 30$ min?

[4]The model can be used to for general *heat transfer* problems, not just *cooling*! It is generally a very good approximation, though there are exceptions when the heat transfer is primarily through radiation, like the transfer of heat from the sun to the earth, or from the heating element in an oven. One of the best applications is for home heating. How much heat is lost through the walls of a house during winter? How much fuel is saved by adding insulation in the walls?

2. The mass $M(t)$ of a radioactive isotope remaining after t years is given by $M(t) = 5\,e^{-0.005t}$ grams.

   (a) What is the mass remaining after

   (i) $t = 0$ hours (ii) $t = 6$ months ?

   (b) How long would it take for the mass to decay to 1 gram ?

   (c) What is the rate of radioactive decay at

   (i) $t = 1$ year (ii) $t = 100$ years ?

   (d) Show that $M'(t) = 0.005 \times M(t)$

3. Diabetics with type 1 diabetes are unable to produce insulin, which is needed to process glucose. These diabetics must injection medications containing insulin that are designed to release insulin slowly. The insulin itself breaks down quickly.

   The decay rate varies greatly between individuals, but the following model shows a typical pattern of insulin breakdown. Here I represents the units of insulin in the bloodstream, and t is the time since the insulin entered the bloodstream in minutes.

$$I = 10\,e^{-0.05t}$$

   (a) explain what the value 10 tells about the amount of insulin in the bloodstream.

   (b) What is the rate of breakdown in insulin in the bloodstream at time t?

4. A population grows according to the model $P(t) = P(0)e^{rt}$ where time t is in years.

   (a) Show that the growth rate $\dfrac{dP}{dt}$ is proportional to $P(t)$.[5]

   (b) Show that the growth rate *per head of population* is r.

5. Show that Newton's Law of Cooling implies that the rate of change of the temperature of an object is proportional to the difference between the object's temperature and the temperature of the environment, that is

$$\frac{dT}{dt} \propto (T(t) - T_{env})\,.$$

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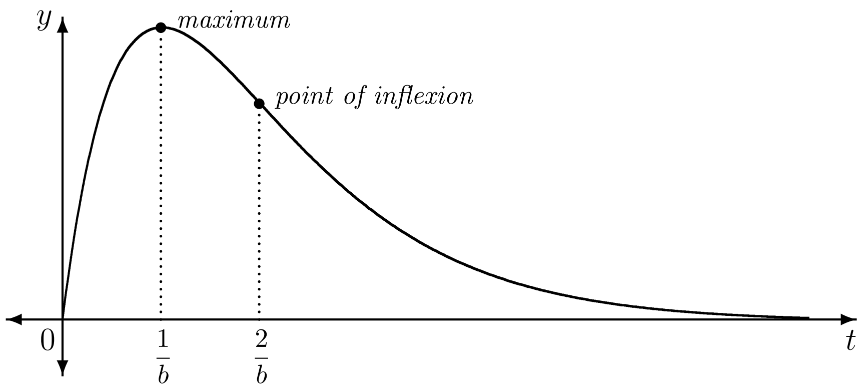
[5]Two quantities, Y and X are said to be proportional, in symbols $Y \propto X$, if Y is equal to a constant multiple of X. The constant is called the *constant of proportionality*.

3.3 Surge models

Surge models have the form

$$y = At\,e^{-bt}, t \geq 0$$

for constants A and b, where independent variable t usually represents *time*.[6]



Surge models are used in Pharmacokinetics to model the uptake of medication. There is a rapid increase in concentration in the bloodstream after introduction by ingestion, injection, or other means, then a slow elimination through excretion or metabolism.

Exercise 3.3

1. After an aspirin tablet is ingested, the amount entering the bloodstream is modelled by $M(t) = 100te^{-0.5t}$ mg, t hours after its absorption into the bloodstream has begun.

   (a) How much aspirin is in the bloodstream after

   (i) $t = 0$ hour (ii) $t = 1$ hour (iii) $t = 2$ hours?

   (b) When is the amount of aspirin in the bloodstream a maximum, and what is the maximum?

   (c) What is the point of inflection of the graph of $M(t) = 100te^{-0.5t}$. What is the significance of this point?

2. The amount of aspirin entering the bloodstream is modelled closely by

$$M(t) = Ate^{-bt} \text{ mg},$$

   t hours after initial absorption into the bloodstream, where A and b can be varied according to the type of tablet and amount of aspirin used.

[6] This is a special case of the general Makoid-Banakar model in which the amount of dissolved drug at time t is given by

$$d(t) = At^n e^{-bt}, \text{ where } A, n, b > 0.$$

What should the values of A and b be if the the maximum amount of aspirin in the blood was 120 gm at $t = 2$ hours?

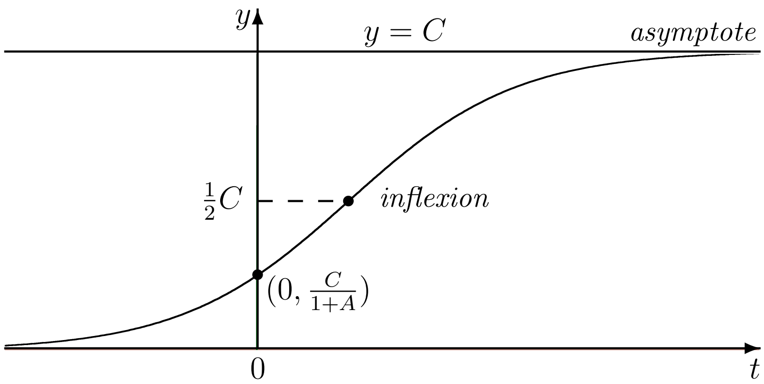
3. What is the turning point of the curve $y = Ax^n e^{-bx},$ where $A, n, b, x > 0$?

3.4 Logistic models

Logistic models have the form

$$y = \frac{C}{1 + A\,e^{-bt}}\ ,\ t \geq 0$$

for constants A, b and C, where independent variable t usually represents *time*.



Logistic models are used to model self-limiting populations where growth is restricted by competition for limited resources.[7] The number C is called the *carrying capacity* of the population.

Exercise 3.4

1. The population of a new colony of bees after t months is given by

$$P(t) = \frac{50\,000}{1 + 1000e^{-0.5t}}$$

   (a) What is the initial population of the colony?
   (b) What is the carrying capacity of the colony.
   (c) How long will it take the population to reach 40 000?
   (d) Show that $P'(t) \geq 0$ for all $t \geq 0$, and interpret this.
   (e) Find when the population growth rate is greatest.

2. Show that the logistic function

$$y = \frac{C}{1 + A\,e^{-bt}}\ ,\ t \geq 0$$

   has a point of inflexion at $\left(\frac{\ln A}{b}, \frac{C}{2}\right)$.

[7]The logistic function was discovered by Pierre F. Verhulst in 1838, and is also called the Verhulst equation. The shape of the graph is sometimes referred to as S-curve or a Sigmoid curve.

Appendix A

Answers

Exercise 1.2

1(a) $5e^x$

1(b) $7e^{7x}$

1(c) $-100e^{-5x}$

1(d) $-2\,\exp(-2x)$

1(e) $10e^{5x}$

1(f) $12x^2 + 10 - e^x$

1(g) $12e^{4x} + 4x$

1(h) $5(e^x - e^{-x})$

1(i) $9e^{3x} + 6e^{2x}$

1(j) $-e^{-x} + e^x - 2e^{-2x}$

1(k) $-6e^{-2x}$

1(l) $-5e^{-x}$

2(a) $(x+1)e^x$

2(b) $(2x - x^2)e^{-x}$

2(c) $\dfrac{1+4x}{2\sqrt{x}}\,e^{2x}$

2(d) $\dfrac{2(x-1)}{x^2}\,e^x$

2(e) $\dfrac{-e^{-x}}{(1-e^{-x})^2}$

2(f) $\dfrac{2e^x}{(e^x+1)^2}$

3(a) $6e^{2x}(e^{2x}+1)^2$

3(b) $\dfrac{-e^{-x}}{2\sqrt{1+e^{-x}}}$

3(c) $\dfrac{-e^{2x}}{(1+e^{2x})^{3/2}}$

3(d) $\dfrac{2+2e^x+xe^x}{2\sqrt{1+e^x}}$

3(e) $2(x+1)e^{(x+1)^2}$

3(f) $\dfrac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

4. $y'(0) = 20\ln 3$

6(a) turning point $(0,0)$; global minimum

6(b) turning point $(-1/2, -1/2\,e^{-1})$; global miminum

6(c) turning point $(0,0)$; global maximum

6(d) turning point $(1, 2e^{-2})$; local minimum

Exercise 2.1

1(a) $\ln 5$	1(b) *not possible*	1(c) $\frac{1}{2}\ln 7$
1(d) $-\ln 0.1$ or $\ln 10$	1(e) $\frac{1}{2}\ln 16$ or $\ln 4$	1(f) $\ln 2$
1(g) 0 or $\ln 2$	1(h) $\ln 2$	1(i) 0
2(a) -1 or 2	2(b) 6	

Exercise 2.2

1(a) $\dfrac{5}{x}$	1(b) $\dfrac{1}{x}$	1(c) $\dfrac{20}{x}$
1(d) $\dfrac{1}{x}$	1(e) $\dfrac{2}{x}$	1(f) $12x^2 + x - \dfrac{1}{x}$
2(a) $\ln x + 1$	2(b) $(2\ln x + 1)x$	2(c) $\dfrac{\ln(2x) + 2}{2\sqrt{x}}$
2(d) $\dfrac{(x\ln x + 1)e^x}{x}$	2(e) $\dfrac{2(1 - \ln x)}{x^2}$	2(f) $\dfrac{\ln x - 1}{2\ln^2 x}$
3(a) $\dfrac{3\ln^2 x}{x}$	3(b) $\dfrac{1}{2x\sqrt{\ln x}}$	3(c) $\dfrac{1}{x+1}$
3(d) $\dfrac{2x}{x^2+1}$	3(e) $\dfrac{1}{x} + \dfrac{2x}{x^2+1}$	3(f) $\dfrac{2x}{x^2+1} + \dfrac{2x}{x^2+2} + \dfrac{2x}{x^2+3}$

Exercise 3.2

1a(i) 5000	1a(ii) 5471	1a(iii) 7167
1(b) 6.1 hours	1c(i) 900 bacteria/hour	1c(ii) 985 bacteria/hour
2a(i) 5 gm	2a(ii) 4.99 gm	2(b) 322 years
2c(i) 0.025 gm/year	2c(ii) 0.015 gm/year	
3(a) initial amount	3(b) $0.5e^{-0.05t}$ units/min	

Exercise 3.3

1a(i) 0 mg 1a(ii) 60.7 gm 1a(iii) 73.6 gm

1b(i) 2 hours 1b(ii) 200/e mg

1c(i) 4 hours 1c(ii) rate of elimination is greatest

2(i) b = 0.5 2(ii) A = 60e

3. $\left[\frac{n}{b}, A\left(\frac{n}{b}\right)^n e^{-n}\right]$

Exercise 3.4

1(a) 49 or 50 1(b) 50 000 1(c) ≈ 16.6 months

1d(i) $P'(t) = \dfrac{25000e^{-0.5t}}{(1+10e^{-0.5t})^2} > 0$ 1d(ii) P(t) is increasing

1(e) $t = 13.8$ months